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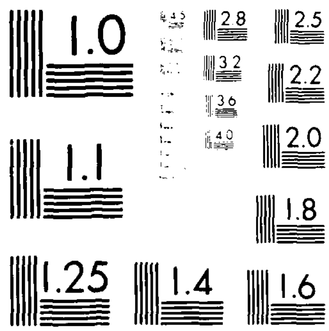

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The following result is proved. If S_n is a separable metric space for $n\le\infty$, $\phi_n: S_n\to S_\infty$ is measurable for $n<\infty$, X_n is an S_n-valued random variable for $n\le\infty$ and $\phi_n(X_n)\to_d X_\infty$ in S_∞, then there exist S_n-valued random variables X_n^* such that $X_n=_d X_n^*$ for $n\le\infty$ and $\phi_n(X_n^*)\to X_\infty^*$ wp1. Conditions on S_n and ϕ_n are presented that allow a construction in the context of Polish spaces.

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STRONG REPRESENTATION OF WEAK CONVERGENCE

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Technical Report No. 186

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STRONG REPRESENTATION OF WEAK CONVERGENCE

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The following result is proved. If S_n is a separable metric space for $n \leq \infty$, $\varphi_n: S_n \to S_\infty$ is measurable for $n < \infty$, X_n is an S_n-valued random variable for $n \leq \infty$ and $\varphi_n(X_n) \to_d X_\infty$ in S_∞, then there exist S_n-valued random variables X_n^* such that $X_n =_d X_n$ for $n \leq \infty$ and $\varphi_n(X_n^*) \to X_\infty^*$ wp1. Conditions on S_n and φ_n are presented that allow a construction in the context of Polish spaces.

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Skorohod's representation theorem • strong representation of weak convergence

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In this paper we prove the following variant of Skorohod's representation theorem for weak convergence. Equality in distribution is denoted by $=_d$, convergence in distribution by $\to_d$.

Theorem 1. *Let S_n for $n = 1,2,\ldots,\infty$ be a separable metric space and let φ_n for $n = 1,2,\ldots$ be a measurable function from S_n into S_∞. If X_n is an S_n-valued random variable for $n = 1,2,\ldots,\infty$ and $\varphi_n(X_n) \to_d X_\infty$ in S_∞, then there exist S_n-valued random variables X_n^* for $n = 1,2,\ldots,\infty$ defined on one probability space and such that $X_n^* =_d X_n$ in S_n for $n = 1,2,\ldots,\infty$ and $\varphi_n(X_n^*) \to X_\infty^*$ wp1 in S_∞.*

When $S_n = S$ (separable) and $q_n = \mathrm{id}_S$ for all n, then the above theorem specializes to Dudley's (1968) variant of Skorohod's representation theorem. In Skorohod's (1956) original version S was required to be complete as well. See Wichura (1970) and Blackwell & Dubins (1983) for further extensions. Our proof of the present theorem amounts to the construction of a special metric space T to which Dudley's theorem can be applied.

Theorem 1 turns out to be useful in many instances. It is applied in Bai (1984), Bai & Yin (1986) and Yin (1984). The need for these applications led the first two authors to the present research.

Here is the simplest example of a theorem that can be proved by Theorem 1, but not by the theorem of Skorohod-Dudley in its original form. It is Theorem 4.1 of Billingsley (1968), restricted to separable metric spaces.

Theorem 2. *If S is a separable metric space with metric ρ, (X_n, Y_n) are S^2-valued random variables for $n = 1, 2, \ldots$ and X is an S-valued random variable such that $X_n \to_d X$ in S and $\rho(X_n, Y_n) \to_d 0$ in* R, *then $Y_n \to_d X$ in S.*

Proof. By Billingsley (1968, Th.4.4) we have $(X_n, \rho(X_n, Y_n)) \to_d (X, 0)$ in $S \times \mathbf{R}$. Apply Theorem 1 with $S_\infty := S \times \mathbf{R}$, $S_n := S^2$, X_n replaced by (X_n, Y_n) and $q_n(x, y) = (x, \rho(x, y))$, all for $n < \infty$. □

Proof of Theorem 1. All statements involving n are supposed to hold for $n = 1, 2, \ldots, \infty$ unless restricted explicitly; limit statements without explicit tendency hold as $n \to \infty$. Let T be the disjoint union of all S_n. Let $s : T \to \{1, 2, \ldots, \infty\}$ be defined by $s(x) = n$ if $x \in S_n$. Set $q_\infty := \mathrm{id}_{S_\infty}$ and define $q : T \to S_\infty$ by $q(x) := q_{s(x)}(x)$. Let ρ_n be the metric of S_n. Let ε_n be positive for $n < \infty$, decreasing to 0 as $n \to \infty$, and set $\varepsilon_\infty := 0$. We now define what is going to be the metric on T:

$$(1) \qquad \delta(x, y) := \rho_\infty(q(x), q(y)) + \begin{cases} \varepsilon_{s(x)} \wedge \rho_{s(x)}(x, y) & \text{if } s(x) = s(y), \\ \varepsilon_{s(x)} \vee \varepsilon_{s(y)} & \text{if } s(x) \neq s(y). \end{cases}$$

Let us first verify that δ is indeed a metric. Obviously, $\delta(x, y) = \delta(y, x)$ and $\delta(x, x) = 0$. If $\delta(x, y) = 0$, then $s(x) = s(y)$ and $\rho_{s(x)}(x, y) = 0$, so $x = y$. The triangle inequality can be verified separately for both terms on the right-hand side of (1). We note the following properties of δ-convergence:

(2) $\delta = \rho_\infty$ on $S_\infty \times S_\infty$.

(3) If $x, x_k \in S_n$, then $\delta(x, x_k) \to 0$ as $k \to \infty$ iff $\rho_n(x, x_k) \to 0$ and $\rho_\infty(q_n(x), q_n(x_k)) \to 0$ as $k \to \infty$. It follows that the ρ_n-topology in S_n is coarser than or equal to the trace in S_n of the δ-topology in T, which is homeomorphic via $S_n \ni x \mapsto (x, q_n(x))$ to the trace in the graph of q_n of the product topology in $S_n \times S_\infty$. The last topology is separable, and so is the trace of the δ-topology in each S_n. Consequently, δ is separable.

(4) S_n is δ-open for $n < \infty$.

(5) If $x_n \in S_n$ for each $n < \infty$, then $x_n \to x$ iff $x \in S_\infty$ and $q_n(x_n) \to x$ in S_∞.

Having established that T with δ is a separable metric space, we may apply Dudley's theorem to T-valued random variables. However, there is one more barrier to take. We want to identify S_n-valued random variables with T-valued random variables having range in S_n. In the first appearance random variables must be $\mathcal{S}_n$-measurable, where $\mathcal{S}_n$ is the Borel field in S_n generated by ρ_n. In the

second appearance they must be $\mathcal{T}_n$-measurable, where $\mathcal{T}_n$ is the trace in S_n of $\mathcal{T}$, the Borel field in T generated by δ. So we must prove $\mathcal{S}_n = \mathcal{T}_n$.

From the second clause in (3) it follows that $\mathcal{S}_n \subset \mathcal{T}_n$. For the converse inclusion we must do a little more. First note that $\mathcal{T}$ is already generated by the open δ-balls in T, since δ is separable. This can be phrased equivalently by stating that $\mathcal{T}$ is the smallest σ-field in T which makes the functions $\delta(x,\cdot)$ measurable for all $x \in T$. Consequently, $\mathcal{T}_n$ is the smallest σ-field in S_n which makes the functions $\delta(x,\cdot)$, restricted to S_n, measurable for all $x \in T$. First suppose $n = \infty$. Then $\delta(x,y) = \rho_\infty(q_\infty(x),y) + \varepsilon_{s(x)}$ for $x \in T$, $y \in S_\infty$, which, as a function of y, is obviously $\mathcal{S}_\infty$-measurable. So $\mathcal{T}_\infty \subset \mathcal{S}_\infty$. Considering (1) for $x \in T$ as a function of $y \in S_n$ we observe that $\rho_n(x,\cdot)$ is $\mathcal{S}_n$-measurable, and that $\rho_\infty(q_\infty(x), q_{\infty n}(\cdot))$ is $\mathcal{S}_n$-measurable as composition of the $\mathcal{S}_n$–$\mathcal{S}_\infty$-measurable function $q_{\infty n}$ and the $\mathcal{S}_\infty$-measurable function $\rho_\infty(q_\infty(x),\cdot)$.

We now write down the scheme of implications that proves the theorem. We are given S_n-valued random variables X_n such that

(6) $\qquad q_{\infty n}(X_n) \to_d X_\infty \quad$ in S_∞.

The major point, to be proved below, is that this implies

(7) $\qquad X_n \to_d X_\infty \quad$ in T.

By Dudley's theorem there are T-valued random variables Y_n, defined on one probability space, such that $Y_n =_d X_n$ in T and $Y_n \to Y_\infty$ wp1 in T. By (4) we have $S_n \in \mathcal{T}$ for each n, so there is a measurable function $j_n: T \to S_n$ such that the restriction of j_n to S_n is the identity map on S_n; take j_n to be the identity map on S_n and constant on $T \setminus S_n$. Set $X_n^* := j_n Y_n$. Then X_n^* has range in S_n, and $X_n^* = Y_n$ wp1 in T, since $\mathsf{P}[Y_n \in S_n] = \mathsf{P}[X_n \in S_n] = 1$. From $X_n^* = Y_n$ wp1 and $Y_n =_d X_n$ in T it follows that $X_n^* =_d X_n$ in T. As X_n^* has range in S_n, this implies $X_n^* =_d X_n$ in S_n. From $X_n^* = Y_n$ wp1 in T and $Y_n \to Y_\infty$ wp1 in T we obtain $X_n^* \to X_\infty^*$ wp1 in T. As X_n^* has range in S_n, this implies by (5)

$$q_{\infty n}(X_n^*) \to X_\infty^* \text{ wp1} \quad \text{in } S_\infty .$$

We have arrived at all conclusions of the theorem.

It remains to prove the implication (6) $\Rightarrow$ (7). We will interpret (6) and (7) by convergence of probability distributions on continuity sets, so we must compare the boundaries under ρ_∞ and δ. By (2) we have for $A \subset T$:

(8) $\qquad \partial_{\rho_\infty}(A \cap S_\infty) = \partial_{\delta|S_\infty}(A \cap S_\infty) \subset \partial_\delta(A)$.

Let $B(x,\varepsilon) := \{y \in T: \delta(x,y) < \varepsilon\}$ and set

(9) $\qquad \mathcal{V} := \{B(x,\varepsilon): \varepsilon < \varepsilon_{s(x)} \text{ if } s(x) < \infty,\ \mathsf{P}[X_\infty \in \partial_\delta B(x,\varepsilon)] = 0\}$,

and let $\mathcal{U}$ consist of the unions of finitely many elements of $\mathcal{V}$. By Billingsley (1968, Corollary 2 on p 15) it is sufficient for (7) that

(10) $\qquad \mathsf{P}[X_n \in A] \to \mathsf{P}[X_\infty \in A] \quad$ for $A \in \mathcal{U}$.

If $B(x,\varepsilon) \in \mathcal{V}$ with $s(x) < \infty$, then $B(x,\varepsilon) \subset S_{s(x)}$, so $\mathsf{P}[X_n \in B(x,\varepsilon)] = 0$ unless $n = s(x)$. If $B(x,\varepsilon) \in \mathcal{V}$ with $x \in S_\infty$, then

$$[X_n \in B(x,\varepsilon)] = [\varrho_\infty(x, \varphi_n(X_n)) < \varepsilon - \varepsilon_n] = [\varphi_n(X_n) \in B(x, \varepsilon - \varepsilon_n) \cap S_\infty],$$

so

$$(11) \qquad [\varphi_n(X_n) \in B(x,\varepsilon) \cap S_\infty] \subset \liminf [X_n \in B(x,\varepsilon)] \subset \limsup [X_n \in B(x,\varepsilon)]$$
$$\subset [\varphi_n(X_n) \in \overline{B(x,\varepsilon) \cap S_\infty}].$$

By (6), (8) and (9) the outmost sides of (11) have equal probabilities. Combining the previous observations for separate $B(x,\varepsilon) \in \mathcal{V}$ we arrive at (11) with $A \in \mathcal{U}$ instead of $B(x,\varepsilon) \in \mathcal{V}$, again with equal probabilities for the outmost sides. This proves (10), hence (7). The proof of the theorem is complete.

Remarks. In general the space T is not complete under δ, even if all S_n are under ϱ_n. To see this, consider the case that all x_n $(n < \infty)$ lie in S_m for one fixed m. Then (x_n) is δ-Cauchy iff $((x_n, \varphi_m(x_n)))_{n<\infty}$ is $\varrho_m \times \varrho_\infty$-Cauchy. If the latter holds, then $((x_n, \varphi_m(x_n)))$ converges in $S_m \times S_\infty$, but not necessarily in graph φ_m, unless the latter is closed. This combined with the observation that δ-Cauchy sequences $(x_n)_{n<\infty}$ with $x_n \in S_n$ converge if S_∞ is ϱ_∞-complete leads us to the following result.

Theorem 3. *Let S_n be separable and ϱ_n-complete for each n. Then T is δ-complete iff* graph φ_n *is closed in $S_n \times S_\infty$ for each n.*

It is well-known that graph φ_n is closed if φ_n is continuous, and that φ_n is continuous if graph φ_n is closed and S_∞ is compact. Using the fact that a subset of a Polish space is Polish iff it is G_δ (Dugundji (1966, Th.XIV.8.3)), we arrive at the following variation on Theorem 3.

Theorem 4. *Let S_n be Polish for each n. Then T is Polish iff* graph φ_n *is G_δ in $S_n \times S_\infty$ for each n.*

For results on real functions with G_δ graphs, see van Rooij & Schikhof (1982, Exerc. 11.Y,Z). Functions of the first class of Baire (pointwise limits of continuous functions) have G_δ graphs. F_σ graphs are also G_δ.

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